RULED SURFACES WITH GENERATORS IN ONE-TO-ONE CORRESPONDENCE*

BY

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1. Introduction

The configuration composed of two ruled surfaces whose generators are in one-to-one correspondence occurs frequently in geometry. If we assume that corresponding generators are not coplanar, we find that we can base a projective theory of this configuration on a system of four ordinary linear first order differential equations in four dependent variables. If we choose the fundamental curves of reference on the two surfaces suitably, we are able to reduce this system of equations to a relatively simple canonical form.

The curves which are fundamental for our canonical form we have called intersector curves. They are in some respects similar to the curved asymptotics on a ruled surface. We also define other curves which are analogous to flecnode curves and have some of their properties.

As an application of our theory, we have employed it to investigate Green-reciprocal ruled surfaces. The method proves to be a fruitful one. We are able to generalize some well known theorems concerning ruled surfaces of congruences Γ and Γ' which are reciprocal in the sense of G. M. Green. And we discover a geometrical characterization of the directrix congruences of a surface in terms of simple concepts.

2. The differential equations

Let the four homogeneous coördinates $y^{(1)}, \ldots, y^{(4)}$ of an arbitrary point P_y on a curve C_y be given as analytic functions of a single independent variable x. And in like manner let the four coördinates of each of three other points P_z, P_ρ, P_σ be given as functions of the same variable x. Let us join P_y and P_z by a straight line l_{yz}; let us also join P_ρ and P_σ by a line $l_{\rho\sigma}$. Then, as x varies, the locus of l_{yz} is a ruled surface R_{yz}, and the locus of $l_{\rho\sigma}$ is a ruled surface $R_{\rho\sigma}$. It is these ruled surfaces that we wish to study.

We shall suppose that P_y, P_z, P_ρ, P_σ are not coplanar, so that corresponding generators l_{yz} and $l_{\rho\sigma}$ do not intersect. Then the determinant

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$$\begin{vmatrix} y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \varrho^{(1)} & \varrho^{(2)} & \varrho^{(3)} & \varrho^{(4)} \\ \sigma^{(1)} & \sigma^{(2)} & \sigma^{(3)} & \sigma^{(4)} \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$(1)\qquad \begin{aligned} y' &= c_{11}\, y + c_{12}\, z + a_{11}\, \varrho + a_{12}\, \sigma, \\ z' &= c_{21}\, y + c_{22}\, z + a_{21}\, \varrho + a_{22}\, \sigma, \\ \varrho' &= b_{11}\, y + b_{12}\, z + d_{11}\, \varrho + d_{12}\, \sigma, \\ \sigma' &= b_{21}\, y + b_{22}\, z + d_{21}\, \varrho + d_{22}\, \sigma \end{aligned}$$

so that $(y^{(i)}, z^{(i)}, \varrho^{(i)}, \sigma^{(i)})$ $(i = 1, 2, 3, 4)$ will be four sets of solutions. For example, we may substitute each of these four sets in turn in the first equation of system (1) and then solve the resulting four equations for the coefficients of the first of equations (1). Similarly we may determine the coefficients of each of the other three equations.

System (1) is not uniquely determined when R_{yz} and $R_{\rho\sigma}$ are given. If we introduce new reference curves $C_{\bar{y}}$ and $C_{\bar{z}}$ on R_{yz} by the transformation

$$\begin{aligned} y &= a\bar{y} + b\bar{z}, \\ z &= c\bar{y} + d\bar{z}, \qquad D \equiv ad - bc \neq 0, \end{aligned}$$

and new reference curves $C_{\bar{\rho}}$ and $C_{\bar{\sigma}}$ on $R_{\rho\sigma}$ by the transformation

$$\begin{aligned} \varrho &= \alpha\bar{\varrho} + \beta\bar{\sigma}, \\ \sigma &= \gamma\bar{\varrho} + \delta\bar{\sigma}, \qquad \Delta \equiv \alpha\delta - \beta\gamma \neq 0, \end{aligned}$$

we do not change the surfaces R_{yz} and $R_{\rho\sigma}$; but system (1) goes over into another system of the same form whose coefficients, indicated by dashes, are given by the following equations:

$$
\begin{aligned}
D\bar{c}_{11} &= d(-a'+c_{11}a+c_{12}c)+b(c'-c_{21}a-c_{22}c),\\
D\bar{c}_{12} &= d(-b'+c_{11}b+c_{12}d)+b(d'-c_{21}b-c_{22}d),\\
D\bar{a}_{11} &= d(a_{11}\alpha+a_{12}\gamma)-b(a_{21}\alpha+a_{22}\gamma),\\
D\bar{a}_{12} &= d(a_{11}\beta+a_{12}\delta)-b(a_{21}\beta+a_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
D\bar{c}_{21} &= -c(-a'+c_{11}a+c_{12}c)-a(c'-c_{21}a-c_{22}c),\\
D\bar{c}_{22} &= -c(-b'+c_{11}b+c_{12}d)-a(d'-c_{21}b-c_{22}d),\\
D\bar{a}_{21} &= -c(a_{11}\alpha+a_{12}\gamma)+a(a_{21}\alpha+a_{22}\gamma),\\
D\bar{a}_{22} &= -c(a_{11}\beta+a_{12}\delta)+a(a_{21}\beta+a_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{b}_{11} &= \delta(b_{11}a+b_{12}c)-\beta(b_{21}a+b_{22}c),\\
\Delta\bar{b}_{12} &= \delta(b_{11}b+b_{12}d)-\beta(b_{21}b+b_{22}d),\\
\Delta\bar{d}_{11} &= \delta(-\alpha'+d_{11}\alpha+d_{12}\gamma)+\beta(\gamma'-d_{21}\alpha-d_{22}\gamma),\\
\Delta\bar{d}_{12} &= \delta(-\beta'+d_{11}\beta+d_{12}\delta)+\beta(\delta'-d_{21}\beta-d_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{b}_{21} &= -\gamma(b_{11}a+b_{12}c)+\alpha(b_{21}a+b_{22}c),\\
\Delta\bar{b}_{22} &= -\gamma(b_{11}b+b_{12}d)+\alpha(b_{21}b+b_{22}d),\\
\Delta\bar{d}_{21} &= -\gamma(-\alpha'+d_{11}\alpha+d_{12}\gamma)-\alpha(\gamma'-d_{21}\alpha-d_{22}\gamma),\\
\Delta\bar{d}_{22} &= -\gamma(-\beta'+d_{11}\beta+d_{12}\delta)-\alpha(\delta'-d_{21}\beta-d_{22}\delta).
\end{aligned}
$$

By a suitable choice of reference curves, the geometrical characterization of which will be furnished later, we are able to reduce system (1) to a simple canonical form. To this end we observe that if we choose a and c as a pair of solutions of the simultaneous differential equations

$$a' = c_{11}a+c_{12}c, \qquad c' = c_{21}a+c_{22}c,$$

we shall then have $\bar{c}_{11} = \bar{c}_{21} = 0$. Likewise we can make $\bar{c}_{12} = \bar{c}_{22} = 0$ by choosing b and d as solutions of

$$b' = c_{11}b+c_{12}d, \qquad d' = c_{21}b+c_{22}d.$$

We can make $\bar{d}_{11} = \bar{d}_{21}$ by choosing α and γ as solutions of

$$\alpha' = d_{11}\,\alpha + d_{12}\,\gamma, \qquad \gamma' = d_{21}\,\alpha + d_{22}\,\gamma,$$

and we can make $\bar{d}_{12} = \bar{d}_{22} = 0$ by choosing β and δ as solutions of

$$\beta' = d_{11}\,\beta + d_{12}\,\delta, \qquad \delta' = d_{21}\,\beta + d_{22}\,\delta.$$

When these reductions have been made, system (1) takes the canonical form

$$\begin{aligned} y' &= a_{11}\,\varrho + a_{12}\,\sigma, & \varrho' &= b_{11}\,y + b_{12}\,z, \\ z' &= a_{21}\,\varrho + a_{22}\,\sigma, & \sigma' &= b_{21}\,y + b_{22}\,z. \end{aligned} \tag{2}$$

Wilczynski has developed the theory of a single ruled surface, for which his fundamental system of equations has the form*

$$\begin{aligned} y'' + p_{11}\,y' + p_{12}\,z' + q_{11}\,y + q_{12}\,z &= 0, \\ z'' + p_{21}\,y' + p_{22}\,z' + q_{21}\,y + q_{22}\,z &= 0. \end{aligned} \tag{A}$$

In order to obtain system (A) from system (1) it is sufficient to differentiate the first two equations of system (1) and then eliminate $\varrho, \sigma, \varrho', \sigma'$. We prefer however to start from our canonical system (2). We find then that the coefficients of system (A), for our surface R_{yz}, are given by the following formulas in terms of the coefficients of system (2):

$$\begin{aligned} p_{11} &= (a_{21}\,a_{12}' - a_{22}\,a_{11}')/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{11} &= -(a_{11}\,b_{11} + a_{12}\,b_{21}), \\ p_{12} &= (a_{12}\,a_{11}' - a_{11}\,a_{12}')/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{12} &= -(a_{12}\,b_{12} + a_{12}\,b_{22}), \\ p_{21} &= (a_{21}\,a_{22}' - a_{22}\,a_{21}')/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{21} &= -(a_{21}\,b_{11} + a_{22}\,b_{21}), \\ p_{22} &= (a_{12}\,a_{21}' - a_{11}\,a_{22}')/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{22} &= -(a_{21}\,b_{12} + a_{22}\,b_{22}). \end{aligned} \tag{3}$$

We are thus enabled to avail ourselves of the classical theory of a single ruled surface.

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* Wilczynski, *Projective Differential Geometry*, Leipzig, 1906, p. 126.

3. GREEN-RECIPROCAL RULED SURFACES

G. M. Green's reciprocal relation R plays a significant part in projective differential geometry. This relation may be briefly formulated as follows.* Consider an arbitrary non-developable surface S. At an arbitrary point on this surface there is a tangent plane and an osculating quadric. A line l which lies in the tangent plane but does not pass through the point of contact, and a line l', which passes through the point of contact but does not lie in the tangent plane, are said to be reciprocal to each other in case they are reciprocal polars with respect to the osculating quadric. The totality of lines l forms a congruence, called a Γ congruence, and the reciprocal lines l' form the reciprocal Γ' congruence.

Let us draw a curve C on our fundamental surface S. Corresponding to every point of this curve we have a pair of reciprocal lines l and l'. All these lines l' form a ruled surface, which intersects S in C, and the reciprocal lines l also form a ruled surface. Such ruled surfaces we shall call Green-reciprocal ruled surfaces.

The generators of Green-reciprocal ruled surfaces are in a one-to-one correspondence and are skew to each other. It will therefore be possible to apply our general theory of pairs of ruled surfaces to Green-reciprocal ruled surfaces. For this purpose we shall need to formulate the relation R analytically.

Let the four homogeneous coördinates $y^{(1)}, \ldots, y^{(4)}$ of an arbitrary point P_y on a surface S_y be given as analytic functions of two independent variables u and v. Let us suppose that S_y is non-degenerate and non-developable, and is referred to its asymptotic net. Then the four functions y are a fundamental system of solutions of a completely integrable system of partial differential equations which may be reduced to the form†

$$(4) \qquad y_{uu}+2by_v+fy = 0, \qquad y_{vv}+2a'y_u+gy = 0.$$

The points P_ρ and P_σ defined by

$$(5) \qquad \varrho = y_u-\beta y, \qquad \sigma = y_v-\alpha y,$$

where α and β are functions of u and v, lie in the tangent plane of S_y at P_y. More precisely, P_ρ lies on the line tangent at P_y to the asymptotic curve

* Green, Proceedings of the National Academy of Sciences, vol. 3 (1917) pp. 587–592.

† Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), p. 246.

$v=$ const. through P_y, and P_σ lies on the line tangent to the asymptotic $u=$ const. at P_y. The line $l_{\rho\sigma}$ which joins P_ρ and P_σ lies in the tangent plane of S_y at P_y, and does not pass through P_y. The line l_{yz} reciprocal to $l_{\rho\sigma}$ joins P_y to the point P_z defined by*

$$z = y_{uv} - \alpha y_u - \beta y_v. \tag{6}$$

An arbitrary curve C_y on S_y (except the curves $u=$ const.) may be defined by expressing v as a function of u, in the form $v=v(u)$. As u varies, P_y moves along the curve C_y, and the lines l_{yz} and $l_{\rho\sigma}$ generate two Green-reciprocal ruled surfaces, R_{yz} and $R_{\rho\sigma}$.

We are now ready to set up the system (1) for Green-reciprocal ruled surfaces. We first calculate the partial derivatives of z, ϱ and σ with respect to u, and with respect to v, obtaining

$$\begin{aligned} z_u &= Py - \beta z + A\varrho - F'\sigma, \\ z_v &= Qy - \alpha z - G'\varrho + B\sigma, \end{aligned} \tag{7}$$

where we have placed

$$\begin{aligned} A &= 4a'b - \alpha_u - \alpha\beta, \qquad F' = f + \beta^2 + \beta_u - 2b\alpha + 2b_v, \\ B &= 4a'b - \beta_v - \alpha\beta, \qquad G' = g + \alpha^2 + \alpha_v - 2a'\beta + 2a_u, \\ P &= A\beta - F'\alpha + 2bg - f_v + f\alpha, \\ Q &= B\alpha - G'\beta + 2a'f - g_u + g\beta; \end{aligned} \tag{8}$$

and obtaining

$$\begin{aligned} \varrho_u &= -Fy - \beta\varrho - 2b\sigma, \qquad \varrho_v = (\alpha\beta - \beta_v)y + z + \alpha\varrho, \\ \sigma_v &= -Gy - 2a'\varrho - \alpha\sigma, \qquad \sigma_u = (\alpha\beta - \alpha_u)y + z + \beta\sigma, \end{aligned} \tag{9}$$

where we have placed

$$F = f + \beta^2 + \beta_u + 2b\alpha, \qquad G = g + \alpha^2 + \alpha_v + 2a'\beta. \tag{10}$$

* Green, *Memoir on the general theory of surfaces and rectilinear congruences,* these Transactions, vol. 20 (1919), p. 87.

We next calculate the total derivatives of y, z, ϱ and σ with respect to u, using the formula $y' = y_u + v' y_v$, where v' is obtained by differentiating the equation $v = v(u)$ defining R_{yz} and $R_{\rho\sigma}$. We obtain in this way a system of equations of the form (1), whose coefficients have the following values:

$$
\begin{array}{llll}
c_{11} = \beta + v'\alpha, & c_{12} = 0, & a_{11} = 1, & a_{12} = v', \\
c_{21} = P + v'Q, & c_{22} = -\beta - v'\alpha, & a_{21} = A - v' G, & a_{22} = -F' + v'B, \\
d_{11} = -\beta + v'\alpha, & d_{12} = -2b, & b_{11} = -F + v'(\alpha\beta - \beta_v), & b_{12} = v', \\
d_{21} = -2v'a', & d_{22} = \beta - v'\alpha, & b_{21} = \alpha\beta - \alpha_u - v'G, & b_{22} = 1.
\end{array} \tag{11}
$$

We shall make use of this system of equations whenever we wish to apply our general theory to Green-reciprocal ruled surfaces.

4. Intersector curves

A curve on R_{yz} will be called an *intersector curve* (with respect to $R_{\rho\sigma}$) in case the tangent at each point of the curve intersects the line $l_{\rho\sigma}$ which corresponds to the generator l_{yz} that passes through the point. A similar definition may be made for intersector curves on $R_{\rho\sigma}$.

We shall now determine the intersector curves on R_{yz}. Any point P_φ on a generator l_{yz} (except P_z) may be defined by setting $\varphi = y + \lambda z$, where λ is a function of x. As x varies, the locus of P_φ is a curve C_φ on R_{yz}. We wish to determine λ so that C_φ will be an intersector curve. The point φ', where $\varphi' = y' + \lambda z' + \lambda' z$, is a point on the tangent of C_φ. Substituting from the first two equations of system (1), we find

$$
\varphi' = (c_{11} + \lambda c_{21}) y + (\lambda' + c_{12} + \lambda c_{22}) z + (a_{11} + \lambda a_{21}) \varrho + (a_{12} + \lambda a_{22}) \sigma. \tag{12}
$$

Then an arbitrary point on the tangent of C_φ is given by an expression of the form $\varphi' + k\varphi$. If this tangent intersects $l_{\rho\sigma}$, there must exist a value of k so that the expression $\varphi' + k\varphi$ is a linear combination of ϱ and σ only. Therefore k and λ must satisfy the two conditions

$$
c_{11} + \lambda c_{21} + k = 0, \qquad \lambda' + c_{12} + \lambda c_{22} + k\lambda = 0.
$$

Eliminating k, we obtain the differential equation of the intersector curves on R_{yz},

(13) $$\lambda' = -c_{12} + (c_{11} - c_{22})\lambda + c_{21}\lambda^2.$$

The corresponding equation for the intersector curves on $R_{\rho\sigma}$ may be deduced in the same way. If we denote an arbitrary point on $l_{\rho\sigma}$ by P_ψ, where $\psi = \varrho + \mu\sigma$, we find

(14) $$\psi' = (b_{11} + \mu b_{21})y + (b_{12} + \mu b_{22})z + (d_{11} + \mu d_{21})\varrho + (\mu' + d_{12} + \mu d_{22})\sigma.$$

And the differential equation for the intersector curves on $R_{\rho\sigma}$ is

(15) $$\mu' = -d_{12} + (d_{11} - d_{22})\mu + d_{21}\mu^2.$$

We observe that if our fundamental equations are written in the canonical form (2), then the curves $\lambda =$ const. are the intersector curves on R_{yz}, and the curves $\mu =$ const. are the intersector curves on $R_{\rho\sigma}$. In fact, inspection of system (1) will show that C_y is an intersector curve if $c_{12} = 0$, and C_z is an intersector curve if $c_{21} = 0$. Similarly, C_ρ is an intersector curve if $d_{12} = 0$, and C_σ is an intersector curve if $d_{21} = 0$.

We remark further that equation (13) is a Riccati equation. Therefore *there is a one-parameter family of intersector curves on R_{yz}, and they are determined by solving a Riccati equation.* Conversely, we can show that, if we have given a ruled surface R_{yz} and a Riccati equation

$$\lambda' = P + Q\lambda + R\lambda^2,$$

then there exists a ruled surface $R_{\rho\sigma}$ with respect to which the one-parameter family of curves defined by the given equation are intersector curves. Comparing the given equation with equation (13), let us choose $c_{11} = 0$, and set

$$c_{12} = -P, \qquad c_{22} = -Q, \qquad c_{21} = R.$$

If we introduce these values into the first two equations of system (1), and solve these two equations for ϱ and σ, we obtain

$$\varrho = Pz + y', \qquad \sigma = -Ry + (P + Q)z + y' + z'.$$

These formulas determine the required ruled surface $R_{\rho\sigma}$. Therefore we have shown that *the theory of a Riccati equation is equivalent to the theory of the intersector curves on a ruled surface with respect to some associated ruled surface.* It should be noted that our determination of $R_{\rho\sigma}$ is not unique, since only the difference $c_{11} - c_{22}$ is defined.

Since the cross ratio of any four particular solutions of a Riccati equation is constant, we see that, *if we select any four intersector curves on R_{yz}, they will cut each generator l_{yz} in a set of four points having the same cross ratio on all generators.*

The curved asymptotics on a ruled surface have long been known* to possess the cross ratio property which we have just established for intersector curves in general. We shall find a Riccati equation defining the curved asymptotics on R_{yz}, using our canonical system (2). Let P_φ, where $\varphi = y + \lambda z$, be an arbitrary point on l_{yz}. Then differentiating twice and making use of system (2), we find

$$
(16)\qquad
\begin{aligned}
\varphi' &= \lambda' z + (a_{11} + \lambda a_{21})\varrho + (a_{12} + \lambda a_{22})\sigma, \\
\varphi'' &= (\,)y + (\,)z + [(a_{11} + \lambda a_{21})' + \lambda' a_{21}]\varrho + [(a_{12} + \lambda a_{22})' + \lambda' a_{22}]\sigma,
\end{aligned}
$$

the omitted coefficients being immaterial. The locus C_φ of P_φ will be an asymptotic curve on R_{yz} if the osculating plane of C_φ at every point coincides with the tangent plane of R_{yz} at the point. Then the four points y, φ, φ', φ'' will be coplanar. The condition for their coplanarity reduces to

$$
(17)\qquad
\begin{aligned}
2(a_{12}a_{21} - a_{11}a_{22})\lambda' &= a_{11}a'_{12} - a_{12}a'_{11} \\
&\quad + (a_{11}a'_{22} - a_{22}a'_{11} + a_{21}a'_{12} - a_{12}a'_{21})\lambda + (a_{21}a'_{22} - a_{22}a'_{21})\lambda^2.
\end{aligned}
$$

This equation defines the curved asymptotics on R_{yz}.

A second property of the asymptotics may also be extended to intersector curves. Since an asymptotic tangent intersects three consecutive generators on a ruled surface, it follows that the asymptotic tangents of R_{yz} constructed at points of a fixed generator l_{yz} form a quadric. This quadric is called the osculating quadric of R_{yz} at l_{yz}. Now a tangent of an intersector curve on R_{yz} intersects two consecutive generators at l_{yz} and also intersects $l_{\rho\sigma}$. Therefore the intersector tangents of R_{yz} constructed at points of a fixed generator

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* P. Serret, *Théorie Nouvelle Géométrique et Mécanique des Lignes à Double Courbure*, Paris, 1860.

l_{yz} also form a quadric, which we shall call the *intersector quadric* of R_{yz} at l_{yz}. In order to find the equation of this quadric, we observe that an arbitrary point on the tangent of an arbitrary intersector curve is given by $\varphi' + k\varphi$, where φ' has the value written in equation (12), and λ satisfies equation (13). If we introduce a local tetrahedron of reference with vertices at the points y, z, ρ, σ, and with suitably chosen unit point, we may write the local coördinates of the point $\varphi' + k\varphi$ in the form

$$x_1 = c_{11} + \lambda c_{21} + k, \qquad x_3 = a_{11} + \lambda a_{21},$$
$$x_2 = (c_{11} + \lambda c_{21} + k)\lambda, \qquad x_4 = a_{12} + \lambda a_{22}.$$

Eliminating k and λ, and making the result homogeneous in the usual way, we obtain the equation of the intersector quadric of R_{yz} in the form

$$a_{12} x_1 x_3 - a_{11} x_1 x_4 + a_{22} x_2 x_3 - a_{21} x_2 x_4 = 0.$$

The relation of asymptotics to intersector curves is made clear by the following considerations. Suppose that $l_{\rho\sigma}$ happens to be a generator of the same set as l_{yz} on the osculating quadric of R_{yz}. Then $R_{\rho\sigma}$ is called a derivative ruled surface* of R_{yz}, and the intersector curves are the asymptotics. *The asymptotics on R_{yz} are intersector curves with respect to an arbitrary derivative ruled surface of R_{yz}.*

5. Analogues of flecnode curves

When four skew straight lines are given, there exist two other straight lines each of which intersects all four given lines. Indeed, the flecnodes on a generator of a ruled surface have been defined† as the two points at each of which a line may be drawn intersecting four consecutive generators. We wish to extend this notion of flecnode in two directions.

In the first place, we recall that an asymptotic tangent of R_{yz} intersects three consecutive generators l_{yz}. Therefore there are two points on each generator l_{yz} which are characterized by the fact that the asymptotic tangent at each of them intersects the corresponding line $l_{\rho\sigma}$. If $l_{\rho\sigma}$ happens to be also a generator of R_{yz}, and if we let $l_{\rho\sigma}$ approach l_{yz} over the surface R_{yz}, then our two points approach the flecnodes of l_{yz} as limiting positions.

* Wilczynski, *Projective Differential Geometry*, p. 147.
† Wilczynski, *Projective Differential Geometry*, p. 149.

We may determine our two points as follows. Let the fundamental equations be written in the canonical form (2), and let any point P_φ on l_{yz} be defined by $\varphi = y + \lambda z$. If the asymptotic tangent at P_φ coincides with the intersector tangent, then equation (17) of the asymptotics and the equation $\lambda' = 0$ of the intersector curves may be regarded as simultaneous. Therefore we determine the two points at which the asymptotic tangents are also intersector tangents by solving the quadratic

$$(18) \quad a_{11}\,a'_{12} + a_{12}\,a'_{11} + (a_{11}\,a'_{22} - a_{22}\,a'_{11} + a_{21}\,a'_{12} - a_{12}\,a'_{21})\,\lambda \\ + (a_{21}\,a'_{22} - a_{22}\,a'_{21})\,\lambda^2 = 0.$$

The locus of each of these points is a curve analogous to a flecnode curve. At each of its points the asymptotic and intersector tangents coincide. This analogue of the flecnode curve is indeterminate in case the asymptotic and intersector curves coincide, that is, in case $R_{\rho\sigma}$ is a derivative of R_{yz}.

We arrive at our second analogue of a flecnode as follows. Since any tangent of R_{yz} intersects two consecutive generators l_{yz}, and since any tangent of $R_{\rho\sigma}$ intersects two consecutive generators $l_{\rho\sigma}$, it follows that there are two lines which are tangent to both R_{yz} and $R_{\rho\sigma}$, the points of contact of each line being on corresponding generators l_{yz} and $l_{\rho\sigma}$.

In order to determine the common tangent lines of R_{yz} and $R_{\rho\sigma}$, we first determine the tangent plane of each surface. The tangent plane of R_{yz} at a point P_φ, where $\varphi = y + \lambda z$, is determined by P_y, P_z and $P_{\varphi'}$, where φ' is given by equation (12). Therefore this plane is determined by l_{yz} and the point

$$(19) \qquad (a_{11} + \lambda a_{21})\,\rho + (a_{12} + \lambda a_{22})\,\sigma,$$

where the plane intersects the generator $l_{\rho\sigma}$ corresponding to l_{yz}. Similarly, the tangent plane of $R_{\rho\sigma}$ at a point P_ψ, where $\psi = \rho + \mu\sigma$, is determined by P_ρ, P_σ, and $P_{\psi'}$, where ψ' is given by equation (14). Therefore this plane is determined by $l_{\rho\sigma}$ and the point

$$(20) \qquad (b_{11} + \mu b_{21})\,y + (b_{12} + \mu b_{22})\,z,$$

where the plane intersects the generator l_{yz} corresponding to $l_{\rho\sigma}$.

Now the line of intersection of the tangent planes of R_{yz} and $R_{\rho\sigma}$ joins the points (19) and (20). This line is tangent to R_{yz} if the point (20) coincides with P_φ, that is, if λ and μ satisfy the condition

$$(21)\qquad \lambda = (b_{12} + \mu b_{22})/(b_{11} + \mu b_{21}).$$

Similarly, this line is tangent to $R_{\rho\sigma}$ if the point (19) coincides with P_ψ, that is, if λ and μ satisfy the condition

$$(22)\qquad \mu = (a_{12} + \lambda a_{22})/(a_{11} + \lambda a_{21}).$$

If we regard equations (21) and (22) as simultaneous and eliminate μ, we obtain

$$(23)\quad (a_{21} b_{11} + a_{22} b_{21}) \lambda^2 + (a_{11} b_{11} + a_{12} b_{21} - a_{22} b_{22} - a_{21} b_{12}) \lambda - (a_{11} b_{12} + a_{12} b_{22}) = 0.$$

Solution of this quadratic determines the two points on l_{yz} at which lines can be drawn tangent to both of R_{yz} and $R_{\rho\sigma}$. The points of contact of these common tangent lines on $R_{\rho\sigma}$ are determined by eliminating λ from equations (21) and (22) and solving the quadratic

$$(24)\quad (a_{11} b_{21} + a_{21} b_{22}) \mu^2 + (a_{11} b_{11} + a_{21} b_{12} - a_{12} b_{21} - a_{22} b_{22}) \mu - (a_{12} b_{11} + a_{22} b_{12}) = 0.$$

6. Applications to Green-reciprocal ruled surfaces

We shal' now consider a pair of Green-reciprocal ruled surfaces and apply our general theory to them. For this purpose we employ equations (11). Since $c_{12} = 0$ in equations (11), it follows that *C_y is an intersector curve on R_{yz}*. Moreover, this fact is geometrically obvious, since the line tangent to C_y lies in the tangent plane of S_y and therefore intersects $l_{\rho\sigma}$. The equation (13) of the intersector curves on R_{yz} becomes

$$(25)\qquad \lambda' = 2(\beta + v'\alpha)\lambda + (P + v'Q)\lambda^2.$$

Since a particular solution of this equation, namely $\lambda = 0$, is known, *the intersector curves on R_{yz} can be determined by quadratures.* In fact, if we place $\lambda = 1/\nu$, equation (25) reduces to the linear equation

$$(26) \qquad \nu' = -(P+v'Q) - 2(\beta+v'\alpha)\nu.$$

We might expect C_y to be also an asymptotic on R_{yz}. But if C_y is an asymptotic on R_{yz}, then the osculating plane of C_y is tangent to R_{yz} at P_y and therefore contains the generator l_{yz} through P_y. Then, using the language of Miss Sperry, we may say that *the curve C_y is an asymptotic on R_{yz} if, and only if, C_y is a union curve* of the congruence Γ'.* This theorem is a generalization of one of Green's theorems.† He considered an arbitrary conjugate net on S_y and the axis congruence of this net. Then his theorem asserts that the ruled surface of axes which intersects S_y in a curve of the fundamental conjugate net has this curve for an asymptotic. To derive Green's theorem from ours, we have only to note that a curve of the fundamental conjugate net is a union curve of the axis congruence of the net.

If we select any particular point on l_{yz}, the plane of this point and $l_{\rho\sigma}$ will touch $R_{\rho\sigma}$ in a definite point. Let us consider the point P_y. The plane of P_y and $l_{\rho\sigma}$, which is the tangent plane of S_y at P_y, touches $R_{\rho\sigma}$ in a point which is of particular interest. Let this point be P_ψ, where $\psi = \varrho+\mu\sigma$. Substituting from equations (11) into equations (20), we find that the tangent plane of $R_{\rho\sigma}$ at P_ψ intersects l_{yz} in the point

$$(27) \qquad [-F+v'(\alpha\beta-\beta_v)+\mu(\alpha\beta-\alpha_u-v'G)]\,y+(v'+\mu)\,z.$$

This point coincides with P_y if $\mu = -v'$. Therefore the tangent plane of S_y at P_y, touches $R_{\rho\sigma}$ at the point $\varrho-v'\sigma$. But if we recall that the surface S_y is referred to its asymptotic net, we see that the point $\varrho-v'\sigma$ lies on the conjugate of the tangent to the curve C_y that corresponds to $R_{\rho\sigma}$. Therefore *the point of contact of the tangent plane of S_y with an arbitrary ruled surface $R_{\rho\sigma}$ lies on the conjugate of the tangent to the curve on S_y that corresponds to $R_{\rho\sigma}$.*

This theorem is a generalization of a theorem which has had an interesting history. It was first proved by Wilczynski for the directrix congruences, and by Sullivan for the scroll directrix congruences. Green proved it for an arbi-

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* P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface,* American Journal of Mathematics, vol. 40 (1918), pp. 213–224.

† G. M. Green, Second Memoir, American Journal of Mathematics, vol. 38 (1916), p. 303.

trary pair of reciprocal congruences. His theorem* amounts to this, that, for a ruled surface $R_{\rho\sigma}$ which is developable, the line $l_{\rho\sigma}$ touches the edge of regression of the developable in a point which lies on the conjugate of the tangent to the curve on S_y which corresponds to the developable. Our generalization is suggested by the fact that the point where $l_{\rho\sigma}$ touches the edge of regression of $R_{\rho\sigma}$, namely the focal point of $l_{\rho\sigma}$, is also the point where the tangent plane of S_y touches $R_{\rho\sigma}$. In fact, any plane containing $l_{\rho\sigma}$ will touch $R_{\rho\sigma}$ at the focal point of $l_{\rho\sigma}$.

When $R_{\rho\sigma}$ is developable the point (27) is indeterminate, and we have

$$(28) \qquad \begin{aligned} v' + \mu &= 0, \\ -F + v'(\alpha\beta - \beta_v) + \mu(\alpha\beta - \alpha_u - v'G) &= 0. \end{aligned}$$

If we eliminate μ we obtain Green's equation for determining the developables of the Γ congruence. And if we eliminate v', we obtain Green's equation for determining the focal sheets of the Γ congruence.†

The locus of the point P_ψ, where $\psi = \rho - v'\sigma$, is a curve on $R_{\rho\sigma}$. We shall call this curve the *contact curve* on $R_{\rho\sigma}$, because at each of its points the tangent plane of S_y touches $R_{\rho\sigma}$. Let us enquire under what condition the contact curve C_ψ is also an intersector curve. Since C_ψ is an intersector curve its tangent at P_ψ must intersect l_{yz}, but, since C_ψ is the contact curve, its tangent must lie in the tangent plane of S_y at P_y. Therefore the tangent passes through P_y and is the conjugate of the tangent of the curve C_y which corresponds to $R_{\rho\sigma}$. The locus of all these conjugate tangents is a developable, called the conjugate developable of C_y, which, in our case, has C_ψ for its edge of regression. So at every point P_y the conjugate of the tangent of C_y touches the edge of regression of the conjugate developable of C_y at a point on $l_{\rho\sigma}$. Such a curve C_y has been called an adjoint union curve‡ of the Γ congruence. Therefore, *if the contact curve on $R_{\rho\sigma}$ is also an intersector curve, then the curve C_y which corresponds to $R_{\rho\sigma}$ is an adjoint union curve of the Γ congruence.* If we wish to prove this theorem analytically, we may substitute from equation (11) into equation (15) and replace therein μ by $-v'$. The result is the well known differential equation of the second order for the adjoint union curves on S_y.

* Green, *Memoir on the general theory of surfaces*, these Transactions, vol. 20 (1919), p. 94.

† Green, loc. cit., p. 89 and p. 91.

‡ Green, loc. cit., p. 140.

Let us again employ the geometrical argument to demonstrate still another theorem concerning the contact curve on $R_{\rho\sigma}$. The tangent planes of S_y, constructed at points of the curve C_y, are also the tangent planes of $R_{\rho\sigma}$ constructed at points of the contact curve. These planes envelope the conjugate developable of C_y, and, if the contact curve is also an intersector curve, the conjugate developable is the developable of the tangents of the contact curve. Since the planes enveloping a developable are also the osculating planes of the edge of regression of the developable, it follows in our case that at each point of the contact curve the osculating plane of the curve is the tangent plane of $R_{\rho\sigma}$. Therefore, *if the contact curve is an intersector curve, then it is an asymptotic.* This theorem may also be proved analytically by direct calculation.

7. The directrix congruences

One of the most important examples of pairs of congruences reciprocal with respect to a surface is the pair of directrix congruences* of the surface. Wilczynski defined the directrix congruences in terms of the osculating linear complexes of the asymptotic curves on the surface. We are able to give another geometrical characterization of them. Let us substitute from equations (11) into equation (24). We obtain in this way the equation

$$\begin{aligned}(29)\qquad &[4a'b-2\alpha_u-v'(G+G')]\mu^2+[F'-F-v'^2(G'-G)]\mu\\ &+v'[F+F'-v'(4a'b-2\beta_v)]=0,\end{aligned}$$

which determines the two points $l_{\rho\sigma}$ at which tangents to $R_{\rho\sigma}$ can be drawn which also touch R_{yz} at points on the corresponding generator l_{yz}. Now these two points separate P_ρ and P_σ harmonically if, and only if,

$$F'-F-v'^2(G'-G)=0.$$

This equation is satisfied for every v', that is, for every pair of Green-reciprocal ruled surfaces, if, and only if,

$$F'=F,\qquad G'=G.$$

* Wilczynski, *Curved surfaces,* Second Memoir, these Transactions, vol. 9 (1908), pp. 79–120.

Reference to equations (8) and (10) will show that these equations are equivalent to

$$\alpha = b_v/2b, \qquad \beta = a'_u/2a'.$$

Therefore the congruences Γ and Γ' are the directrix congruences.* Consequently *we may characterize the directrix congruences by saying that they are the only reciprocal congruences which have the property that, for every pair of Green-reciprocal ruled surfaces R_{yz} and $R_{\rho\sigma}$, the two points, on $l_{\rho\sigma}$, at which tangents of $R_{\rho\sigma}$ can be drawn which are also tangent to R_{yz} at points on l_{yz}, separate P_ρ and P_σ harmonically.*

* G. M. Green, *Memoir on the general theory of surfaces,* these Transactions, vol. 20 (1919), p. 92.

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